TOPIC 3
Differential equations. Methods of their integration

3.1. Conception of differential equations

An equation which specifies a relationship between a function, its argument and its derivatives of the first, second, etc. order is called a ***differential equation***. Thus a differential equation could be of the form

$$y'(x) \equiv \frac{dy(x)}{dx} = f(x, y).$$

Classification of differential equations:

1) ordinary - partial

A **differential equation** is called **ordinary** if the unknown function and its derivatives depend only on one independent variable.

In a **partial differential equation** the unknown function and its derivatives depend on at least two independent variables.

Example:

The time-dependent Schrödinger equation,

$$\frac{-h^2}{2m}\frac{\partial^2\phi}{\partial x^2} + i\frac{\partial\phi}{\partial t} + (V(x) - E)\phi(x) = 0$$

is a partial differential equation as it contains two independent variables *x* and *t*.

2) order

The order of a differential equation is the highest n^{th} derivative present in the differential equation.

A **first order differential equation** is an equation that involves *x*, *y*, and $\frac{dy}{dx}$. If $\frac{d^2y}{dx^2}$ also appears in the equation it is called a **second order differential equation**.

So the equation $\frac{dy}{dx} = 4$ is a **first-order differential equation**, whereas the equation $\frac{d^2y}{dx^2} = 4$ is a **second-order differential equation**.

Example:

The equation describing the harmonic oscillator,

$$\frac{d^2x}{dt^2} + \omega_0^2 x = A\sin(\omega t)$$

is a **second-order differential equation** as the highest derivative is of second order.

3) homogenous - inhomogenous

A differential equation is called **homogenous** if every term in it depends on the unknown function or its derivatives. It is **inhomogenous** if there is at least one term which depends only on the independent variables or is a constant different than zero.

Example:

The differential equation of a driven harmonic oscillator,

$$\frac{d^2x}{dt^2} + \omega_0^2 x = 0$$

is inhomogeneous as the term on the right only depends on *t*.

4) linear - nonlinear

A ***linear*** differential equation contains only linear terms of the unknown function and its derivatives.

Example:

The differential equation describing a oscillating pendulum,

$$\frac{d^2\phi}{dt^2} + \frac{g}{l}\sin(\phi) = 0$$

is a common example of nonlinear equation.

Every linear ordinary differential equation is of degree one, but the converse is not true.

Any nonlinear ordinary differential equation (ODE) can be approximated by a linear ODE through a process called *linearization*.

5) degree

The degree of a differential equation is the highest power of the highest order derivative.

Examples.

$y''' + 2(y'')^5 + y = 0$ is an equation of degree 1.

$(y'')^3 + (y')^7 = 0$ is an equation of degree 3.

3.2. First-order differential equations

Type of equation	Method of solution
$\frac{dy}{dx}=f(x)$	Direct integration
$\frac{dy}{dx}=f(x)g(y)$	Separating the variables
Homogeneous	Substitute $y=vx$
Linear equations	Integrating factor

The main problem of the theory of differential equations is to find the unknown function which, substituted into the differential equation, turns it into an identity. Such a function is called the ***solution or integral of the differential equation***.

It is possible that a differential equation has no solution. More usual is that the differential equation has infinitely many solutions. Even the simplest kind of first order differential equation has usually an infinite manifold of solution.

3.2.1. Direct integration

It is the appropriate method if you have a differential equation like:

$$\frac{dy}{dx}=f(x)$$

where $f(x)$ is any function of x (including the possibilities that $f(x)$=constant or $f(x)=0$). The function f is continuous on an open interval I.

> With some simple algebraic manipulations, the differential equation solved - integrate both sides of the equation.

To solve a differential equation means to find a continuous function of the independent variable $y = F(x)$ that, along with its derivatives, satisfies the equation $\frac{dy}{dx}=f(x)$.

Remember that all the antiderivatives of $f(x)$ form a family of functions, which differ from each other by a constant.

This family is just the indefinite integral of f,

$$\int f(x)dx=F(x)+C$$

where C is an constant which is not defined by the differential equation. This is the source of the non-uniqueness of the solution of differential equation: we can give the integration constant any value, and each time we specify a different value of C we are also selecting a different solution of the differential equation.

The family of functions (solutions) is the general solution of the differential equation, which depend on the constant C of the initial condition.

Examples.

1. $\frac{d}{dt}y(t)=t \quad \Rightarrow \quad dy(t)=tdt \quad \Rightarrow \quad \int dy(t)=\int tdt;$

$y(t)=\frac{1}{2}t^2+C.$

2. $\frac{dy}{dx}=2x \quad \Rightarrow \quad dy=2xdx \quad \Rightarrow \quad \int dy=\int 2xdx$

So $y(x)=x^2+C$.

3. $\frac{dy}{dx}=\sin x \quad \Rightarrow \quad dy=\sin xdx \quad \Rightarrow \quad \int dy=\int \sin xdx$

So $y(x)=-\cos(x)+C$.

4. $\frac{dy}{dx}=3e^x \quad \Rightarrow \quad dy=3e^x dx \quad \Rightarrow \quad \int dy=\int 3e^x dx$

So $y(x)=3e^x+C$.

5. $\frac{dy}{dx}=x^{-2}+x+\cos x \quad \Rightarrow \quad dy=(x^{-2}+x+\cos x)dx \quad \Rightarrow$

$\Rightarrow \int dy=\int (x^{-2}+x+\cos x)dx \quad \Rightarrow \quad \int dy=\int x^{-2}dx+\int xdx+\int \cos xdx.$

So $y(x)=-x^{-1}+0.5x^2+\sin(x)+C$.

Boundary conditions

You may recall that we were looking at the equation $\frac{dy}{dx} = 4$ and we solved this to get the solution *y(x)=4x+C*. At the time there were infinitely more solutions to that equation.

Here's one of those other solutions: *y(x)=4x+1*.

Here's another: *y(x)=4x+2*.

And so on.......

These are all solutions because if we differentiate *y(x)=4x+c*, where c is any constant, we find that $\frac{dy}{dx} = 4$, since the constant disappears when you differentiate it.

This is probably familiar to you from integration, whenever we integrated a function we always had to add on a "constant of integration", usually called *C*.

This means that whenever we solve a differential equation (i.e. we integrate it to find the function that satisfies it) we can always add a constant on to the solution.

So if we want to end up with a **unique** solution to a differential equation, **we need another piece of information as well as the equation it satisfies, in order to determine the constant**.

For ***example,*** if we were asked for the solution of the equation $\frac{dy}{dx} = 4$ that satisfies *y(0)=0*, then there would be only one possible answer: *y(x)=4x*.

If we try any of the other solutions to the equation, may be *y(x)=4x+1* or *y(x)=4x+2*, say, we find that *y(0)* is not zero, but 1 or 2 respectively.

Such a condition, which is given in conjunction with a differential equation to fix the constant of integration and so give a unique solution, is called a **boundary condition**.

3.2.2. Separable equations

It is the appropriate method if you have a differential equation like:

$$\frac{dy}{dt} = g(t)h(y)$$

first order differential equation with separable variables.

In an equation with separable variables, $\frac{dy}{dt}$ is a product of a function of *t* and a function of *y*.

In particular, $\frac{dy}{dt} = g(t)$ is a differential equation with separable variables in which *h(y)* is the constant 1. Similarly, $\frac{dy}{dt} = h(y)$ is a differential equation with separable variables in which *g(t)* is the constant 1. An equation with separable variables can be solved by separating the variables and integrating both sides of the equation.

In an equation of this form the variables are said to be *separable.*

We simply separate the *x* and *y* terms (dividing by *h(y)* and multiplying by *dx*) and integrate, we get

$$\int \frac{dy}{h(y)} = \int f(x)dx.$$

Many techniques to solve order differential equations have as their guiding principle some strategy of reducing the equations to a separable form.

Example 1.

$$\frac{dx}{x} = -dt$$

Integrating both sides gives

$$\ln(|x(t)|) = -t + C,$$

where **C** is a constant of integration. Next, we can solve for *x(t)* by exponentiation both sides

$$e^{\ln(|x(t)|)} = e^{-t+C},$$

$$|x(t)| = e^{-t+C} = e^{-t}e^{C}.$$

Finally, we can get rid of the absolute value sign by replacing e^{C}, which is always positive, by a constant **C**, which can be positive or negative. So our solution can be written as

$$x(t) = Ce^{-t}.$$

Initial conditions

The solution $x(t)=Ce^{-t}$ satisfies equation $\frac{dx}{x}=-dt$ for any value of ***C***. Plugging ***t=0*** into our solution tells us that $x(0)=C$. That is, the value of ***C*** is equal to the value of *x(0)*. For example, if we wanted the solution of equation $\frac{dx}{x}=-dt$ that satisfied *x(0)=3*, the answer would be

$x(t)=3e^{-t}$.

An equation of the form *x(0)=K*, where ***K*** is a constant is called an ***initial condition*** for a differential equation. For separable differential equations, it should be clear that the solution process always introduces a constant of integration. An initial condition is needed to determine the value of this constant.

Example 2.

$$\frac{dy}{dx}=(1+x)(2+y)$$

To solve this equation, the first step as before is to move the *y*-term away from the right-hand side on to the left:

$$\frac{1}{2+y}dy=(1+x)dx.$$

The second step is to integrate both sides:

$$\int\frac{1}{2+y}dy=\int(1+x)dx.$$

The integrals are of standard form and we can do them, to obtain the required relation between *x* and *y*. Don't forget the constant of integration.

$$\ln(2+y)=x+\frac{1}{2}x^2+C$$

$$e^{\ln(2+y)}=e^{x+\frac{1}{2}x^2+C}$$

$$2+y=e^{x+\frac{1}{2}x^2}e^C$$

$$y=Ce^{x+\frac{1}{2}x^2}-2.$$

Example 3.

$$\frac{dy}{dx}=\frac{y+xy}{y+2}$$

Sometimes we need to rearrange the right-hand side to get it into the form of a function of *x* times a function of *y*, as in this case. Here we can take a *y* outside a bracket on the top, so that the two variables can be separated:

$$\frac{dy}{dx}=\frac{y(1+x)}{y+2}.$$

Then to solve this equation, we again move the *y*-term away from the right-hand side on to the left:

$$\frac{y+2}{y}dy=(1+x)dx.$$

As before, the next step is to integrate both sides:

$$\int\frac{y+2}{y}dy=\int(1+x)dx.$$

We need to split up the integral on the left-hand side before we can do it:

$$\int\left(\frac{y}{y}+\frac{2}{y}\right)dy=x+\frac{1}{2}x^2+C$$

and then it simplifies into a form we can do:

$$\int\left(1+\frac{2}{y}\right)dy=x+\frac{1}{2}x^2+C.$$

So we obtain the required relation between *x* and *y*. Don't forget the constant of integration.

$$y+2\ln|y|=x+\frac{1}{2}x^2+C.$$

Example 4.

Exponential growth or decay. Let *a* be a constant. The exponential growth or decay equation describes a situation in which a variable grows or shrinks at a rate proportional to the amount present

$$\frac{dy}{dx} = ay .$$

Separate: $\frac{dy}{dx} = ay, \quad \int \frac{dy}{y} = \int a dx$

Integrate and solve for *y*:

$\ln|y| = ax + C, \quad |y| = e^{ax+C} = e^C e^{ax}, \quad y = C_0 e^{ax}.$

We've replaced $\pm e^C$ with C_0. If $a>0$, then *y* increases as *x* increases: <u>exponential growth</u>. If a<0, then *y* decreases as *x* decreases: <u>exponential decay</u>.

Example 5.

$x^2 dx + y(x-1)dy = 0$

Separate:

$$x^2 dx + y(x-1)dy = 0;$$
$$x^2 dx = -y(x-1)dy;$$
$$-\frac{x^2}{x-1}dx = ydy;$$
$$-\int \frac{x^2}{x-1}dx = \int ydy;$$
$$-\int \frac{x^2-1+1}{x-1}dx = \int ydy;$$
$$-\int \frac{(x+1)(x-1)+1}{x-1}dx = \int ydy;$$
$$-\int\left(\frac{(x+1)(x-1)}{x-1} + \frac{1}{x-1}\right)dx = \int ydy;$$

integrate:

$$-\int\left(x+1+\frac{1}{x-1}\right)dx = \int ydy;$$
$$-\left(\frac{1}{2}x^2 + x + \ln|x-1|\right) + C = \frac{1}{2}y^2;$$
$$-x^2 - x - 2\ln|x-1| + C_0 = y^2.$$

Observe that there is one *integration step*, hence only one constant.

In the last line *2C* is replaced with C_0. It would not be wrong to write *2C*, but this is neater. You can always rename constant quantities to make the result look nicer.

Finally, the problem did not include an initial condition; hence, it can stopped at y^2, rather than taking square roots. Without initial condition, it can't tell which square root to take.

3.2.3. Substitute *y=vx*

It is the appropriate method if you have a homogeneous differential equations.

Example 1.

Find the general solution to the following:

$$y' = \frac{y}{x} + \frac{y^2}{x^2}$$

Solution

Fist substituting $v = \frac{y}{x}$, so $y' = v + v^2$

From substitution

$y = vx$

$y' = \frac{dy}{dx} = (vx)' = v'x + vx' = \frac{dv}{dx}x + v = v + v^2$

$$v+x\frac{dv}{dx}=v+v^2$$

$$x\frac{dv}{dx}=v^2$$

then separating variables, we get

$$\frac{dv}{v^2}=\frac{dx}{x}$$

$$\int\frac{dv}{v^2}=\int\frac{dx}{x}$$

$$-\frac{1}{v}=\ln|x|+c\ ,$$

substituting back, we get $y=\frac{-x}{\ln|x|+c}$.

Example 2.

$$y'=\frac{y+x}{x}$$

Solution

Substitution is $v=\frac{y}{x}$, so $y'=v+1$

From substitution

$$y=vx$$

$$y'=\frac{dy}{dx}=(vx)'=v'x+vx'=\frac{dv}{dx}x+v=v+1$$

$$v+x\frac{dv}{dx}=v+1$$

$$x\frac{dv}{dx}=1$$

separating variables, we get

$$dv=\frac{dx}{x}$$

$$\int dv=\int\frac{dx}{x}$$

$$v=\ln|x|+C$$

y=xln|x|+Cx.

Example 3.

$$y'=\frac{x^2+xy+y^2}{x^2}$$

Solution

Fist substituting $v=\frac{y}{x}$, so $y'=1+v+v^2$

From substitution

$$y=vx$$

$$y'=\frac{dy}{dx}=(vx)'=v'x+vx'=\frac{dv}{dx}x+v=1+v+v^2$$

$$v+x\frac{dv}{dx}=1+v+v^2$$

$$x\frac{dv}{dx}=1+v^2$$

$$\frac{dv}{1+v^2}=\frac{dx}{x}$$

$$\arctan v=\ln|x|+C$$

$$v=\tan(\ln|x|+C)$$

y=xtan(ln|x|+C).

3.2.4. Integrating factor

It is the appropriate method if you have linear differential equations.

A differential equation is called *linear* if it contains the unknown function *y(x)* and its derivative $y'(x)$ linearly, i.e. if it is of the form

$$p(x)y' + q(x)y = f(x)$$

where *p(x)*, *q(x)* and $f(x)$ are known functions of *x*.

In the particular case, when $f(x) = 0$, the differential equation is called **homogeneous***;* if $f(x) \neq 0$, the linear differential equation is called **non-homogeneous**.

Consider first the **homogeneous linear differential equation**

$$p(x)y' + q(x)y = 0$$

Multiplying by *dx*, dividing by *p(x)* and rearranging we get

$$\frac{dy}{y} = -\frac{q(x)}{p(x)}dx$$

and hence, integrating, we get

$$\ln y = -\int \frac{q(x)}{p(x)}dx + C'$$

where we have denoted the integration constant by C'. If we now exponentiate and set $C' = \ln C$, we get the general solution in the following form:

$$y = Ce^{\left\{-\int \frac{q(x)}{p(x)}dx\right\}}$$

This result shows that one can always find the general solution of a homogeneous differential equation by quadrature: all that remains to do, given the functions *p(x)* and *q(x)*,is to take the integral of their ratio.

For the **<u>non-homogeneous case</u>**, replace the constant C by a function, and substitute back into the equation to get C.

Examples.

Find the general solutions of the following ***linear homogeneous*** differential equations:

1. $y' + xy = 0$

Solution

$$\frac{dy}{dx} = -xy$$

$$\frac{dy}{y} = -xdx$$

$$\int \frac{dy}{y} = \int -xdx$$

$$\ln y = -\frac{x^2}{2} + C$$

$$y = Ce^{-\frac{x^2}{2}}.$$

2. $xy' + y = 0$

Solution

$$x\frac{dy}{dx} = -y$$

$$\frac{dy}{y} = -\frac{dx}{x}$$

$$\int \frac{dy}{y} = -\int \frac{dx}{x}$$

$$\ln y = -\ln x + \ln C \qquad \Rightarrow \qquad \ln y = \ln \frac{C}{x}$$

$$y = \frac{C}{x}.$$

3. $y' \sin x + y \cos x = 0$

Solution

$$\sin x \frac{dy}{dx} = -y \cos x$$

$$\frac{dy}{y} = -\frac{\cos(x)}{\sin(x)} dx$$

$$\int \frac{dy}{y} = -\int \frac{\cos(x)}{\sin(x)} dx$$

$$\ln y = -\ln \sin x + C$$

$$\underline{y = Ce^{-\ln \sin x}.}$$

Find the general solutions of the following ***linear non-homogeneous*** differential equation

$y' + x^2 y = x^2.$

Solution

To non-homogeneous equation correspond homogeneous

$y' + x^2 y = 0$

its solving is standard

$$\frac{dy}{dx} = -x^2 y$$

Separate:

$$\frac{dy}{y} = -x^2 dx$$

$$\int \frac{dy}{y} = -\int x^2 dx$$

$$\ln y = -\frac{x^3}{3} + \ln C$$

$$e^{\ln y} = e^{-\frac{x^3}{3} + \ln C} = e^{-\frac{x^3}{3}} e^{\ln C}.$$

Solution of homogeneous equation is

$$y = Ce^{-\frac{x^3}{3}}.$$

Now replace the constant C by a function

$$y' = \left(Ce^{-\frac{x^3}{3}}\right)' = \frac{dC}{dx} e^{-\frac{x^3}{3}} + C \frac{d\left(e^{-\frac{x^3}{3}}\right)}{dx}$$

$$y' = \frac{dC}{dx} e^{-\frac{x^3}{3}} + Ce^{-\frac{x^3}{3}} \cdot \left(-x^2\right).$$

Insert y and y' in initial equation

$$\frac{dC}{dx} e^{-\frac{x^3}{3}} + Ce^{-\frac{x^3}{3}} \cdot \left(-x^2\right) + x^2 \cdot Ce^{-\frac{x^3}{3}} = x^2$$

$$\frac{dC}{dx} e^{-\frac{x^3}{3}} = x^2$$

$$dC = x^2 \cdot e^{\frac{x^3}{3}} dx$$

$$\int dC = \int x^2 \cdot e^{\frac{x^3}{3}} dx = \left| \begin{array}{l} t = \dfrac{x^3}{3} \\ dt = t' dx = x^2 dx \\ dx = \dfrac{dt}{x^2} \end{array} \right. = \int e^t dt = e^t + C_1 = e^{\frac{x^3}{3}} + C_1$$

$C = e^{\frac{x^3}{3}} + C_1.$

Insert value of C in homogeneous equation solution.

So, general solution is

$$y = \left(e^{\frac{x^3}{3}} + C_1 \right) e^{-\frac{x^3}{3}} = 1 + C_1 e^{-\frac{x^3}{3}}.$$

Exercises

Independent work in the class

1. Find general and partial solution of the given differential equations:

1. $y' = 5x^3$ if $x = 0, y = 0;$
2. $y' = 3xy$ if $x = 0, y = 1;$
3. $y' = y^2$ if $x = 1, y = 1;$
4. $y' \tan x - y = 1$ if $x = \frac{\pi}{2}, y = 1;$

2. Find the solution of the given differential equations:

1. $y' = y^2 \sin x;$
2. $xyy' = 1 - x;$
3. $e^y y' = 4x^3;$
4. $y' = -x + 2^x;$
5. $\frac{dx}{dt} = x \cos t;$
6. $(x+3)dy - (y+3)dx = 0;$
7. $(x+a)dx = xdy;$
8. $xydx + (x+1)dy = 0;$
9. $2ydx = 3xdy;$
10. $xy' = y \ln x.$

3. Find the solution of the given linear homogeneous differential equations:

1. $2xyy' = y^2 - 4x^2;$
2. $y^2 - 4xy + 4x^2 y' = 0.$

4. ***Find the solution of the given linear non-homogeneous differential equations:***

1. $y' - 6y = e^{3x};$
2. $y' + 2xy = x^2 e^{-x^2};$
3. $y' + y \cos x = \cos x \sin x;$
4. $xy' + y = y^2 \ln x;$

Homework

Find the solution of the given differential equations:

1. $y' = y^2 \cos x;$
2. $(x-3)dy + (y-3)dx = 0;$
3. $\frac{dy}{dx} = e^{x-y};$
4. $y' = t \cdot \sin(t^2);$
5. $y' = e^{-y};$
6. $\frac{dy}{dx} = \frac{x^2}{y};$
7. $y' = y^2 t;$
8. $y' = (1+y^2)e^t;$
9. $y' = y \tan t;$
10. $y' = e^{-3t};$
11. $\frac{dy}{dx} = 2y - 5;$
14. $\frac{dy}{dx} = xe^y;$
15. $xydx - (x-1)dy = 0;$
16. $\frac{dy}{dx} = y^3 \sin x;$
17. $y' = y \sin t;$
18. $y' = y^2 + 1;$
19. $\frac{dy}{dx} = xy^2;$
20. $y' = \sqrt{1-y^2} \cos t;$
21. $y' = \frac{2t+1}{2y-1};$
22. $y' = y^3;$
23. $y' = ty(y+1);$
24. $\frac{dy}{dx} = x^2 y^2 + x^2;$

12. $\frac{dy}{dx} = xy + x + y + 1;$

13. $\sin(y)y' = \sin(x+y) - \sin y \cos x;$

25. $\frac{dy}{dx} = \sqrt{xy};$

26. $dy/dx = 3x^2 - 4x.$

Find the solution of the given linear homogeneous differential equations:

1. $xdy - ydx = ydy;$
2. $xy' = y - xe^{y/x};$
3. $(x-y)dx + (x+y)dy = 0;$
4. $xy' - y = xtg(y/x).$

Find the solution of the given linear non-homogeneous differential equations:

1. $y' - \frac{3}{õ} ó = õ^3;$
2. $y' \cos x = 1 + y \sin x;$
3. $y' + y = x\sqrt{y};$
4. $(x^2+1)y' + 4xy = 3.$

SUPLEMENTARY

Indefinite integrals, its properties. Basic methods of integration

3.1. Conception of indefinite integrals

Integration is the inverse operation of differentiation.
Consider for instance the function

$$f(x) = 2x.$$

A function which has *f(x)* as derivative is

$$F(x) = x^2.$$

We call *F(x)* the **antiderivative of a function *f (x)*** when the derivative of *F(x)* is *f(x):*

$$F'(x) = f(x).$$

The function $F(x) = x^2$ is not the only antiderivative of $f(x) = 2x$. Also the functions $G(x)=x^2+10$ and $H(x)=x^2-5$ are antiderivatives. This is so because additive constants vanish when differentiating. However, two antiderivative can differ only by an additive constant.

Whereas an antiderivative is determined only up to a constant, the following question has a unique answer: $(F(x)+C)' = F'(x) + C' = f(x).$

Example. Find the antiderivative *F(x)* of $f(x)=2x$ with <u>$F(5)=2$ (**initial condition**)</u>.

Solution:

An antiderivative of *f(x)* must be of the form $F(x)=x^2+C$ with a suitable constant C. To have F(5) = 2 we must have

$F(5) = 5^2 + C = 25+C = 2$

$C = 2 - 25 = -23$

Hence $F(x) = x^2 - 23$ solves the problem.

Notation and terminology:

If *F(x)* is an antiderivative of *f(x)* ($F'(x) = f(x)$) one writes

$$\int f(x)dx = F(x) + C$$

e.g. $\int 2xdx = x^2 + C$

and calls $\int f(x)dx$ the ***indefinite integral*** because the result is definite only up to a constant. The function *f(x)* is called the ***integrand*** and C is called the ***constant of integration***. The expression above is read as "the indefinite integral of *f(x)* with respect to *x*."

Rules of integration. Table of integrals

These rules are just inverted rules of differentiation.

1. The equation

$$\int f(x)dx = F(x) + C$$

can be written in the form

$$\int du = u + C \text{ or } \int g'(x)dx = g(x) + C \text{ or } \frac{d}{dx}\int f(x)dx = f(x)$$

The differential symbol *d* and the indefinite integral symbol $\int$ behave as inverses to each other.

2. Constant rule

$$\int cf(x)dx = c\int f(x)dx$$

3. Rule of sums

$$\int((f(x)+g(x))dx = \int f(x)dx + \int g(x)dx$$

Table of integrals

(The constant of integration is left out.)

1. $\int x^r dx = \frac{x^{r+1}}{r+1}$, *r≠-1*	8. $\int \frac{1}{\sin^2 x}dx = -\cot x$
2. $\int \frac{1}{x}dx = \int x^{-1}dx = \ln\|x\|$	9. $\int \frac{dx}{\sqrt{1-x^2}} = \arcsin x = -\arccos x$
3. $\int e^x dx = e^x$	10. $\int \frac{dx}{1+x^2} = \arctan x = -arc\cot x$
4. $\int a^x dx = \frac{a^x}{\ln a}, \quad 0 < a \neq 1$	11. $\int \tan x dx = -\ln\|\cos x\|$
5. $\int \sin x dx = -\cos x$	12. $\int \cot x dx = \ln\|\sin x\|$
6. $\int \cos x dx = \sin x$	13. $\int \frac{1}{\sqrt{a^2-x^2}}dx = \arcsin \frac{x}{a}$
7. $\int \frac{1}{\cos^2 x}dx = \tan x$	14. $\int \frac{1}{a^2+x^2}dx = \frac{1}{a}\arctan \frac{x}{a}$

3.2. Integration methods

They distinguish three simplest integration methods:

- direct integration;
- variable substitution integration;
- integration by parts.

3.2.1. Direct integration

Direct integration is integration when we use algebraic transformations and indefinite integral properties and reduce subintegral expressions to basic integration formulas (table of integrals**)**.

Example 1.

$$\int(5e^x + \sqrt[3]{x^2} + \frac{5}{x})dx = \int(5e^x + x^{\frac{2}{3}} + 5x^{-1})dx = 5\int e^x dx + \int x^{\frac{2}{3}}dx + 5\int x^{-1}dx =$$

$$= 5e^x + C_1 + \frac{3}{5}x^{\frac{5}{3}} + C_2 + 5\ln|x| + C_3 = 5e^x + \frac{3}{5}x^{\frac{5}{3}} + 5\ln|x| + C,$$

where $C = C_1 + C_2 + C_3$.

Example 2.

$$\int(5x\sqrt{x} + 3e^x - 7\cos x)dx = 5\int x^{\frac{3}{2}}dx + 3\int e^x dx - 7\int \cos x dx =$$

$$= 5\frac{x^{\frac{5}{2}}}{\frac{5}{2}} + C_1 + 3e^x + C_2 - 7\sin x + C_3 = 2x^2\sqrt{x} + 3e^x - 7\sin + C,$$

where $C = C_1 + C_2 + C_3$.

3.2.2. Integration by change of variables (Integration by substitution)

In this section we shall show how to make use of the Chain rule for differentiation in problems of integration. The Chain rule will lead to the important method of *integration by change of variables*. *<u>The basic idea</u> is to try to simplify the function to be integrated by changing from one independent variable to another.*

With the above rules we cannot yet integrate simple functions like e^{5x+2} or $\frac{1}{3-2x}$. However, the solution can easily be guessed using the chain rule.

Example 1. $\int e^{5x+2}dx$.

$$\int e^{5x+2}dx = \left| \begin{array}{l} let\ \ u = 5x+2 \\ du = u'dx \\ du = (5x+2)'dx \\ du = 5dx \\ dx = \dfrac{du}{5} \end{array} \right| = \int e^u \frac{du}{5} = \frac{1}{5}\int e^u du = \frac{1}{5}e^u + C = \frac{1}{5}e^{5x+2} + C.$$

Example 2. $\int \frac{1}{3-2x}dx.$

$$\int \frac{1}{3-2x}dx = \left| \begin{array}{l} let\quad u = 3-2x \\ du = (3-2x)'dx \\ du = -2dx \\ dx = -\dfrac{du}{2} \end{array} \right| = -\int \frac{1}{u}\frac{du}{2} = -\frac{1}{2}\ln|u| + C = -\frac{1}{2}\ln|3-2x| + C.$$

Example 3a. $\int xe^{x^2}dx.$

$$\int xe^{x^2}dx = \left| \begin{array}{l} let\quad u = x^2 \\ du = \left(x^2\right)'dx \\ du = 2xdx \\ xdx = \dfrac{du}{2} \end{array} \right| = \int e^u \frac{du}{2} = \frac{1}{2}\int e^u du = \frac{1}{2}e^u + C = \frac{1}{2}e^{x^2} + C.$$

Example 3b. $\int xe^{x^2}dx.$

$$\int xe^{x^2}dx = \left| \begin{array}{l} let\quad u = e^{x^2} \\ du = \left(e^{x^2}\right)'dx \\ du = 2x\cdot e^{x^2}dx \\ x\cdot e^{x^2}dx = \dfrac{du}{2} \end{array} \right| = \int \frac{du}{2} = \frac{1}{2}\int du = \frac{1}{2}u + C = \frac{1}{2}e^{x^2} + C.$$

Example 4. $\int \frac{x}{1+x^2}dx.$

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$$\int \frac{x}{1+x^2}dx = \left| \begin{array}{l} let\quad u = 1+x^2 \\ du = \left(1+x^2\right)'dx \\ du = 2xdx \\ xdx = \dfrac{du}{2} \end{array} \right| = \frac{1}{2}\int \frac{1}{u}du = \frac{1}{2}\ln|u| + C = \frac{1}{2}\ln(1+x^2) + C.$$

Example 5. $\int \tan(x)dx.$

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$$\int \tan xdx = \int \frac{\sin x}{\cos x}dx = \left| \begin{array}{l} let\quad u = \cos x \\ du = (\cos x)'dx \\ du = -\sin xdx \\ \sin xdx = -du \end{array} \right| = \int -\frac{1}{u}du = -\ln|u| + C = -\ln|\cos x| + C.$$

Example 6. $\int e^{1-x}dx$.

$$\int e^{1-x}dx = \left| \begin{array}{l} let \quad u = 1-x \\ du = (1-x)' dx \\ du = -dx \\ dx = -du \end{array} \right| = \int -e^u du = -e^u + C = -e^{1-x} + C.$$

Example 7. $\int x\sqrt{1+x^2}\,dx$.

$$\int x\sqrt{1+x^2}\,dx = \left| \begin{array}{l} let \quad u = 1+x^2 \\ du = (1+x^2)' dx \\ du = 2xdx \\ xdx = \dfrac{du}{2} \end{array} \right| = \int \frac{1}{2}\sqrt{u}\,du = \frac{1}{2}\cdot\frac{2}{3}u^{\frac{3}{2}} + C = \frac{1}{3}(1+x^2)^{\frac{3}{2}} + C.$$

Example 8. $\int \frac{1}{x^2+2x+1}dx$

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The trick here is to rewrite the integrate as a perfect square: $\frac{1}{x^2+2x+1} = \frac{1}{(x+1)^2}$.

$$\int \frac{1}{x^2+2x+1}dx = \int \frac{1}{(x+1)^2}dx = \left| \begin{array}{l} let \quad u = x+1 \\ du = (x+1)' dx \\ du = dx \end{array} \right| = \int \frac{1}{u^2}du = -\frac{1}{u} + C = -\frac{1}{x+1} + C.$$

Example 9. $\int \frac{1}{x^2+2x+2}dx.$

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Guided by our experience with the last example, we will complete the square. This means we will rewrite:

$$x^2+2x+2 = (x^2+2x+1)+2-1 = (x+1)^2+1$$

$$\int \frac{1}{x^2+2x+2}dx = \int \frac{1}{(x+1)^2+1}dx = \left| \begin{array}{l} let \quad u = x+1 \\ du = (x+1)' dx \\ du = dx \end{array} \right| =$$

$$= \int \frac{1}{1+u^2}du = \tan^{-1}(u) + C = \tan^{-1}(x+1) + C.$$

Example 10. $\int \frac{1}{x^2+2x+5}dx$.

This example is pretty similar to the previous one so we'll just show the idea:

$$\int \frac{1}{x^2+2x+5}dx = \int \frac{1}{(x+1)^2+4}dx = \int \frac{1}{4+u^2}du =$$

$$= \frac{1}{4}\int \frac{1}{1+\left(\frac{u}{2}\right)^2}du = \frac{1}{2}\tan^{-1}\left(\frac{u}{2}\right) + C = \frac{1}{2}\tan^{-1}\left(\frac{x+1}{2}\right) + C.$$

Example 11. $\int \cos^2 x dx.$

$$1+\cos 2x = 1+\cos^2 x - \sin^2 x =$$

$$= \sin^2 x + \cos^2 x + \cos^2 x - \sin^2 x = 2\cos^2 x$$

$$\int \cos^2 x dx = \frac{1}{2}\int (1+\cos 2x)dx = \frac{1}{2}\int dx + \frac{1}{2}\int \cos 2x dx =$$

$$= \frac{1}{2}x + \frac{1}{4}\sin 2x + C.$$

After reading the examples above, if you feel that this method requires a certain amount of guessing and luck, then you've got the idea. However, with experience, you will quickly see how to make educated guesses and it will go much more smoothly. Don't be afraid to make some mistakes as you go along, provided you think about what went wrong when you do make a mistake.

Remember that you can always check your answer by differentiating.

3.2.3. Integration by parts

Integration by parts is a useful strategy for simplifying some integrals.

Integration by parts is the inverse to the Product rule

$$(uv)' = u'v + uv'$$

for function *u(x), v(x).* We can rewrite this as $uv' = (uv)' - u'v$ or

$$\boxed{\int uv'dx = uv - \int vu'dx}$$

since *uv* is the antiderivative of *(uv)′*.

$$\boxed{\int udv = uv - \int vdu}$$

Remark. We can see importance of correct applying of rule in integration by parts.

Example 1. The integral $\int xe^x dx$ can be simplified by the identification

1 version	2 version
1) $u = e^x$ $du = de^x = e^x dx$ 2) $v'dx = dv = xdx$ $v = \int dv = \int xdx = \frac{x^2}{2} + C$	1) $u = x$ $du = dx$ 2) $v'dx = dv = e^x dx$ $v = \int dv = \int e^x dx = e^x + C$

1 version. $\int xe^x dx = \frac{x^2}{2}e^x - \int \frac{x^2}{2}e^x dx$

This does not simplify matters. We get more difficult integral.

2 version. The value of the integral is then

$$\int xe^x dx = xe^x - \int e^x dx = xe^x - e^x + C = e^x(x-1) + C \text{ (Answer)}$$

Example 2.

$$\int \ln x dx$$

This seems a bit more confusing since it only looks like there is one "part" to the integration. However, the one part we do see would certainly be easier to deal with if differentiated. Here is the trick to use:

1 version	2 version
1) $u = \ln x$ $du = d\ln x = \frac{1}{x}dx$ 2) $v'dx = dv = dx$ $v = \int dv = \int dx = x + C$	1) $u = 1$ $du = 0$ 2) $v'dx = dv = \ln x dx$ $v = \int dv = \int \ln x dx$ (initial integral)

Then it follows that

$$\int \ln x dx = x\ln x - \int \frac{x}{x}dx = x\ln x - x + C.$$

To solve integrals without mystics study attentive this rule!

LIATE rule

A rule of thumb developed in 1983 for choosing which of two functions is to be *u* is the ***LIATE rule***. According to the rule, whichever function comes first in this list is to be *u*:

L: *logarithmic functions*: *ln x*, $log_2(x)$, etc.

I: *inverse trigonometric functions*: *arctan x, arcsec x*, etc.

A: *algebraic functions*: x^2, $3x^{50}$, etc.

T: *trigonometric functions*: *sin x, tan x*, etc.

E: *exponential functions*: e^x, 13^x, etc.
The function which is to be *dv* by whichever function comes last in the list. You can remember the list by the mnemonic LIATE. The reason for this is that functions lower on the list have easier antiderivatives than the functions above them.

Example 3. $\int x^3 e^{x^2}dx$

$$\int x^3 e^{x^2}dx = \int x^2 \cdot x e^{x^2}dx$$

According LIATE rule

$$\int x^3 e^{x^2}dx = \left| \begin{matrix} u = x^2; & du = \left(x^2\right)' dx = 2xdx; \\ dv = xe^{x^2}dx; & v = \int xe^{x^2}dx = \left| \begin{matrix} let \quad x^2 = t \\ dt = 2xdx \\ xdx = \dfrac{dt}{2} \end{matrix} \right| = \int e^t \dfrac{dt}{2} = \dfrac{1}{2}e^t = \dfrac{1}{2}e^{x^2}. \end{matrix} \right|$$

This results in

$$\int x^3 e^{x^2}dx = x^2 \cdot \frac{1}{2}e^{x^2} - \int \frac{1}{2}e^{x^2} 2xdx = x^2 \cdot \frac{1}{2}e^{x^2} - \frac{1}{2}e^{x^2} + C =$$

$$= \frac{1}{2}e^{x^2}\left(x^2 - 1\right) + C.$$

Example 4. $\int \operatorname{arctan} xdx$

According LIATE rule

$$\int \operatorname{arctan} xdx = \left| \begin{matrix} u = \operatorname{arctan} x & du = \dfrac{1}{1+x^2}dx \\ dv = dx & v = x \end{matrix} \right| = x \cdot \operatorname{arctan} x - \int \frac{x}{1+x^2}dx =$$

$$= x \cdot \operatorname{arctan} x - \frac{1}{2}\ln\left(1 + x^2\right) + C$$

Example 5.

An interesting example is: $\int e^x \cos xdx$ *(by analogy* $\int e^x \sin xdx$)

where, strangely enough, the actual integral does not need to be evaluated.
This example uses integration by parts twice.
First let:

$$\int e^x \cos xdx = \left| \begin{matrix} u = \cos x & du = -\sin xdx \\ dv = e^x dx & v = \int e^x dx = e^x \end{matrix} \right| = e^x \cos x + \int e^x \sin xdx$$

Now, to evaluate the remaining integral, we use integration by parts again,

Then: $\int e^x \sin xdx = \left| \begin{matrix} u = \sin x & du = \cos xdx \\ dv = e^x dx & v = \int e^x dx = e^x \end{matrix} \right| = e^x \sin x - \int e^x \cos xdx$

Putting these together, we get

$$\int e^x \cos xdx = e^x \cos x + e^x \sin x - \int e^x \cos xdx$$

Notice that the same integral shows up on both sides of this equation. So we can simply add the integral to both sides to get:

$$2\int e^x \cos xdx = e^x \cos x + e^x \sin x + c$$

$$\int e^x \cos xdx = \frac{e^x \cos x + e^x \sin x}{2} + C$$

where, *c* (and C = c/2) is an arbitrary constant of integration.

3.3. Particular solutions

We have seen that an integral produces a whole family of solutions parameterized by C. In most applications, we are given an initial or other condition and hence find the value of C. The antiderivative with known C is called ***a particular*** solution.

Example: Find a solution to

$$\int (4x-3)dx$$

given that *f(1) = 2* .

Solution: We first find an antiderivative:

$$\int (4x-3)dx = 2x^2 - 3x + C \; .$$

Now plug in 1 for *x* and 2 for *F* to get:

$$2 = 2(1)^2 - 3(1) + C = -1 + C \, .$$

So that **C = 3**. **The particular solution is**

$$F(x) = 2x^2 - 3x + 3.$$

Exercises

Independent work in the class

1. Find the integral of the following functions:

1. $\int (x+1)(x+2)dx;$
2. $\int \frac{\sin 2x}{\sin x}dx;$
3. $\int \frac{dx}{\sqrt{x}};$
4. $\int \frac{4-x}{2+\sqrt{x}}dx;$
5. $\int 2\sqrt{x}dx;$
6. $\int \frac{x^2-5}{x}dx;$
7. $\int \frac{dx}{x^3};$
8. $\int \frac{e^{2x}+e^x\cos x}{e^x}dx.$

2. Find the integral of the following functions (integration by substitution):

1. $\int \cos 5x dx;$
2. $\int \frac{2xdx}{x^2+1};$
3. $\int \sin x \cos^2 x dx;$
4. $\int \frac{\ln^3 x dx}{x};$
5. $\int e^{\sin x}\cos x dx;$
6. $\int \frac{dx}{\sin^2 2x};$
7. $\int e^{2x+1}dx;$
8. $\int \cos^2 \frac{x}{2}dx;$
9. $\int (e^x + e^{-x})dx;$
10. $\int \frac{\sqrt{1+\ln x}}{x}dx;$
11. $\int \tan x dx;$
12. $\int \frac{dx}{\sqrt{4-25x^2}}.$

3. Find the integral of the following functions (integration by the part):

1. $\int x \ln x dx;$
2. $\int x e^{-x}dx;$
3. $\int x \sin x dx;$
4. $\int \arcsin(x)dx;$
5. $\int x^2 \cos x dx;$
6. $\int e^x \sin x dx;$
7. $\int x^3 \ln x dx.$
8. $\int x^2 e^{-2x}dx.$

Homework

Find the integral of the following functions:

1. $\int \frac{\sin x}{\cos^3 x}dx\,;$
2. $\int \frac{6x}{(x^2+1)^2}dx\,;$
3. $\int x \cos x dx\,;$

25. $\int \frac{dx}{\sin^2 3x}\,;$
26. $\int \frac{\sqrt{\ln x}}{x}dx\,;$
27. $\int x^2 \sin 3x^3 dx\,;$

4. $\int \cos(3x-1)dx;$

5. $\int e^x \sin x dx;$

6. $\int x \ \sin 3x dx$;

7. $\int \frac{\sin x}{\cos^3 x} dx$;

8. $\int \frac{6x}{(x^2+1)^2} dx$;

9. $\int x \sin x dx$;

10. $\int \sin(3x+1)dx;$

11. $\int e^x \cos x dx;$

12. $\int \frac{\cos x}{5+2\sin x} dx$;

13. $\int \arctan x dx;$

14. $\int \frac{x dx}{x^2-5};$

15. $\int x^2 e^{x^3} dx;$

16. $\int \frac{x^3}{\sqrt{1+x^4}} dx;$

17. $\int \frac{dx}{x \ln^2 x}$;

18. $\int \frac{(2+\ln x)^2}{x} dx$;

19. $\int \frac{\cos x}{1+2\sin x} dx$;

20. $\int x(x^2+3)^4 dx$;

21. $\int \cos(8-x)dx$;

22. $\int \frac{e^x dx}{3+4e^x}$;

23. $\int \frac{dx}{x \ln^3 x}$;

24. $\int \cos(ax-b)dx$;

28. $\int e^{5x+4} dx$;

29. $\int e^{\cos x} \sin x dx;$

30. $\int \sin(ax-b)dx$;

31. $\int \frac{dx}{\cos^2 3x}$;

32. $\int \frac{\cos x}{1+5\sin x} dx$;

33. $\int x(x^2+2)^5 dx$;

34. $\int x e^{3x^2-1} dx$;

35. $\int e^{\sin x} \cos x dx;$

36. $\int \frac{e^x dx}{1+e^x}$;

37. $\int x^2 e^{x^3-1} dx;$

38. $\int \frac{x^3}{\sqrt{1+x^4}} dx;$

39. $\int e^x \sqrt{e^x-3} dx;$

40. $\int \frac{x dx}{x^2+3};$

41. $\int \frac{\cos x}{1+2\sin x} dx$;

42. $\int \frac{\sqrt{\ln x}}{x} dx$;

43. $\int \frac{e^x dx}{1+2e^x}$;

44. $\int \sin(5-x)dx$;

45. $\int e^x \sqrt{e^x+1} dx;$

46. $\int \frac{e^x dx}{3+e^x};$

47. $\int \frac{(1+\ln x)^2}{x} dx;$

48. $\int \cot x dx;$